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DIVISORS ON SYMMETRIC PRODUCTS OF CURVES

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ABSTRACT. For a curve with general moduli, the Neron-Severi group of its symmetric products is generated by the classes of two divisors x and θ. In this paper we give bounds for the cones of effective and ample divisors in the $x\theta$-plane.

1. INTRODUCTION

Let C be a smooth irreducible curve of genus g, $J(C)$ its Jacobian variety and C_d its d-fold symmetric product. Fixing a point P_0 on C, we define the maps

$$u_d\colon C_d \to J(C) \quad \text{by} \quad u_d(D) = \mathscr{O}(D - dP_0),$$
$$i_{d-1}\colon C_{d-1} \to C_d \quad \text{by} \quad i_{d-1}(D) = D + P_0.$$

On $J(C)$ we denote by $\boldsymbol{\Theta}$, the theta divisor and by θ, its class in the Neron-Severi group. On C_d we denote by θ_d, or simply (again) θ, the class of $u_d^*(\theta)$ and by x_d, or simply x, the class of $i_{d-1}(C_{d-1})$ in C_d. For curves C with general moduli, it is known that the Neron-Severi group of the symmetric product C_d is generated by θ and x, see [A-C-G-H]. In this paper we give estimates for the cones of the effective and ample divisors on C_d, in the θ, x-plane.

By standard theory, we know the following things about the map u_d:

1. ABEL'S THEOREM. The fiber of the map u_d containing the divisor D, is exactly the set of divisors belonging to the complete series of D.

2. JACOBI'S INVERSION THEOREM. The map $u_g\colon C_g \to J(C)$ is onto.

3. POINCARÉ'S FORMULA. We denote by W_d the image of C_d in the Jacobian by the map u_d and by w_d its class. For $0 \leq d \leq g$ we have

$$w_d = \frac{\theta^{g-d}}{(g-d)!}.$$

In particular, for $d = g - 1$ we have $w_{g-1} = \theta$.

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4. If $d \leq g$, the map u_d is birational to its image, reflecting the fact that $\mathbf{h}^0(C, D) = 1$, for D general point in C_d.
If $g+1 \leq d \leq 2g-2$, the generic fiber of u_d has dimension $d-g$.
If $2g-1 \leq d$, the map is a $\mathbf{P}^{d-g}$-fibration with tautological class x.

Lemma 1 (Intersections on C_d). *For $0 \leq r \leq d \leq g$ we have on C_d*

$$\theta^r x^{d-r} = \frac{g!}{(g-r)!}. \tag{1}$$

Proof. Indeed, $x = \text{class}(i_{d-1}(C_{d-1})) = \text{class}\{D + P_1, D \in C_{d-1}, P_1$ a fixed point in $C\}$ and so,

$$x^{d-r} = \text{class}\{D + P_1 + \cdots + P_{d-r}, D \in C_r, P_1, \ldots, P_{d-r} \text{ fixed in } C\}.$$

Therefore $u_{d*}x^{d-r} = w_r$. Since the map u_d is generically 1-1, projection formula implies that $\theta^r x^{d-r}$ on C_d, is equal to $\theta^r w_r$ on $J(C)$. Poincaré's Formula and the fact that $\theta^g = g!$ complete the proof. Q.E.D.

We denote by $H^{2n}_{an}(C_d, \mathbf{Q})$ the algebraic part of $H^{2n}(C_d, \mathbf{Q})$. Given an algebraic cycle Z in C_d, we define the maps

$$A_k \colon H^{2m}_{an}(C_d, \mathbf{Q}) \to H^{2m}_{an}(C_{d+k}, \mathbf{Q})$$

by

$$A_k(Z) = \{E \in C_{d+k}, E - D \geq 0 \text{ for some } D \in Z\}$$

and

$$B_k \colon H^{2m}_{an}(C_d, \mathbf{Q}) \to H^{2m-2k}_{an}(C_{d-k}, \mathbf{Q})$$

by

$$B_k(Z) = \{E \in C_{d-k}, D - E \geq 0 \text{ for some } D \in Z\}.$$

We have the standard formulas, see [A-C-G-H, p. 367].

Lemma 2 (Push-pull formulas for symmetric products).

$$A_k(x^\alpha \theta^\beta) = \sum_{i=0}^{k} \binom{\beta}{i} \binom{g-\beta+i}{i} \binom{d+k-\alpha-2\beta}{k-i} i! x^{\alpha+i} \theta^{\beta-i} \tag{2}$$

and

$$B_k(x^\alpha \theta^\beta) = \sum_{j=0}^{k} \binom{\alpha}{k-j} \binom{\beta}{j} \binom{g-\beta+j}{j} j! x^{\alpha-k+j} \theta^{\beta-j}. \tag{3}$$

2. Ample and effective divisors

We will use often the following criterion for ampleness, see [Ha]:

Lemma 3 (Nakai-Moishezon). *Let D be a Cartier divisor on a variety X. Then D is ample on X, if and only if, for every subvariety Y in X of dimension r, we have that $D^r Y > 0$.*

In particular, when X is a smooth surface this says that the cone of effective and the cone of ample divisors are dual under the intersection pairing.

If X is a smooth surface, we have the following numerical criterion, for checking effectivity of a divisor D on X, see [Ha]:

Lemma 4. *Let X be a smooth surface, H an ample divisor on X and D a divisor with $D^2 > 0$ and $DH > 0$.*

Then, for n sufficiently large, nD is an effective divisor.

Proof. We prove first that for n large enough we have $\mathbf{h}^2(X, nD) = 0$: Indeed, $\mathbf{h}^2(X, nD) = \mathbf{h}^0(X, K_X - nD)$, and if we assume that $\mathbf{h}^2(X, nD) > 0$ for all n, then $K_X - nD$ is effective and so, $(K_X - nD)H > 0$ (Lemma 3) i.e. $K_X H > nDH$. Since $DH > 0$, taking n big enough—namely $n > K_X H$—we get $K_X H > n$. A contradiction. Therefore there exists an n with $\mathbf{h}^2(X, nD) = 0$ and the Riemann-Roch theorem completes the proof. Q.E.D.

In the higher dimensional case i.e. if X is a smooth variety of dimension d, the Riemann-Roch theorem gives that

$$\begin{aligned}\mathbf{h}^0(X, D) - \mathbf{h}^1(X, D) + \mathbf{h}^2(X, D) - \cdots + (-1)^d \mathbf{h}^d(X, D)\\ = \frac{\mathbf{c}_1^d(D)}{d!} + (\text{terms containing strictly lower powers of } \mathbf{c}_1).\end{aligned}$$

If H is again an ample divisor on X and D a divisor satisfying $D^r H^{d-r} > 0$ for all $0 \le r \le d$, then it is *not* known if

$$\mathbf{h}^0(X, nD) > 0 \quad \text{for } n \text{ big enough}$$

In order to have such a conclusion, we have to impose extra condition, for example, the restriction of $\mathscr{O}(D)$ on H to be an ample divisor. We have

Theorem 1. *Let X be a d-dimensional variety, H an ample effective divisor on X and D a divisor with $D^d > 0$ and $\mathscr{O}(D)|_H$ is an ample line bundle on H.*

Then,

$$\mathbf{h}^0(nD) > 0 \quad \textit{for } n \textit{ big enough}.$$

Proof. We use the following lemmas:

Lemma 5 (Serre). *Let X be a proper scheme over a Noetherian ring. If $\mathscr{N}$ is an invertible sheaf on X, then the following conditions are equivalent*:

(i) *$\mathscr{N}$ is ample.*

(ii) *For each coherent sheaf $\mathscr{C}$ on X, there exists an integer n_0 depending on C s.t. for each $i \ge 1$ and each $n \ge n_0$*

$$\mathbf{H}^i(X, \mathscr{C} \otimes \mathscr{N}^n) = 0.$$

Lemma 6. *Let S be a proper scheme over a Noetherian ring, $\mathscr{L}$ an ample line bundle on S and $\mathscr{F}$ a line bundle generated by global sections. Then there exists an n_0 s.t.*

$$\forall n \ge n_0 \quad \mathbf{H}^i(S, \mathscr{L}^n \otimes \mathscr{F}^k) = 0 \quad \forall k \ge 0, \ \forall i \ge 1.$$

Acknowledgment. I would like to thank J.-F. Burnol, for showing me Lemma 6 above and the following proof.

Proof. We have an exact sequence $\bigoplus_{i=1}^r \mathscr{O}_s \to \mathscr{F} \to 0$. Tensoring by $\check{\mathscr{F}}$ we get $\bigoplus_{i=1}^r \check{\mathscr{F}} \to \mathscr{O}_S \to 0$. Hence (Koszul):

$$0 \to (\check{\mathscr{F}})^r \to \bigoplus_r (\check{\mathscr{F}})^{r-1} \to \bigoplus_{\binom{r}{2}} (\check{\mathscr{F}})^{r-2} \to \cdots \to \bigoplus_r \check{\mathscr{F}} \to \mathscr{O}_S \to 0.$$

And so, tensoring by $\mathscr{F}^r$ we get

$$0 \to \mathscr{O}_S \to \bigoplus_r \mathscr{F} \to \bigoplus_{\binom{r}{2}} (\mathscr{F})^2 \to \cdots \to \bigoplus_r (\mathscr{F})^{r-1} \to (\mathscr{F})^r \to 0.$$

By Lemma 5 we can choose n_0 so that $\mathbf{H}^{\mathbf{i}}(S, \mathscr{L}^n \otimes \mathscr{F}^k) = 0$ for all $i \geq 1$, $n \geq n_0$, $k = 0, 1, \ldots, r$. Then this n_0 works: suppose the claim is true for $k \leq l$. Tensoring the above exact sequence by $\mathscr{L}^n \otimes \mathscr{F}^{l+1-r}$, all sheaves have zero $\mathbf{H}^{\mathbf{i}}$'s, $i \geq 1$, $n \geq n_0$, except maybe the last one which is $\mathscr{L}^n \otimes \mathscr{F}^{l+1}$. Therefore the last one has also zero $\mathbf{H}^{\mathbf{i}}$'s, $i \geq 1$, $n \geq n_0$. Q.E.D.

Going back to the proof of Theorem 1, we use first the following fact: "A line bundle L is ample on H iff L_{red} is ample on H_{red}". By this fact, we can replace H by mH without changing hypothesis on D, and so, we can assume that H is in fact *very* ample on X. We denote by L the line bundle $\mathscr{O}(D)$. We have that $L_1 = L \otimes \mathscr{O}_H$ is ample on H. Let H_1 be the restriction of $\mathscr{O}(H)$ to H; then, H_1 is generated by global sections. Also L_1 is ample on H and so, by the above Lemma 6, there exists an integer n_0 s.t.

$$\mathbf{h}^{\mathbf{i}}(H, L_1^n \otimes H_1^k) = 0 \quad \text{for all } i \geq 1,\ n \geq n_0,\ k \geq 0. \tag{4}$$

On the other hand H is ample on X and so, given the coherent sheaf L^n there exists by Lemma 5 an integer m_n s.t. $\mathbf{H}^{\mathbf{i}}(X, L^n \otimes H^{m_n}) = 0$. Consider now the exact sequence:

$$0 \to \mathscr{O}_X(L^n \otimes H^{(l-1)}) \to \mathscr{O}_X(L^n \otimes H^l) \to \mathscr{O}_H(L_1^n \otimes H_1^l) \to 0.$$

The corresponding long exact sequence gives

$$\begin{aligned}
0 &\to \mathbf{H}^{\mathbf{0}}(X, L^n \otimes H^{(l-1)}) \to \mathbf{H}^{\mathbf{0}}(X, L^n \otimes H^l) \to \mathbf{H}^{\mathbf{0}}(H, L^n \otimes H_1^l) \\
&\to \mathbf{H}^{\mathbf{1}}(X, L^n \otimes H^{(l-1)}) \to \mathbf{H}^{\mathbf{1}}(X, L^n \otimes H^l) \to \mathbf{H}^{\mathbf{1}}(H, L^n \otimes H_1^l) \\
&\to \mathbf{H}^{\mathbf{2}}(X, L^n \otimes H^{(l-1)}) \to \mathbf{H}^{\mathbf{2}}(X, L^n \otimes H^l) \to \mathbf{H}^{\mathbf{2}}(H, L^n \otimes H_1^l) \\
&\cdots \\
&\to \mathbf{H}^{\mathbf{k}}(X, L^n \otimes H^{(l-1)}) \to \mathbf{H}^{\mathbf{k}}(X, L^n \otimes H^l) \to \mathbf{H}^{\mathbf{k}}(H, L^n \otimes H_1^l) \\
&\cdots
\end{aligned}$$

and so, we get for each $n \geq n_0$ that

$$\mathbf{h}^{\mathbf{i}}(X, L^n \otimes H^{(l-1)}) = \mathbf{h}^{\mathbf{i}}(X, L^n \otimes H^l) \quad \text{for all } i \geq 2,\ l \geq 1.$$

Therefore, for each $n \geq n_0$ we have that

$$\mathbf{h}^{\mathbf{i}}(X, L^n) = \mathbf{h}^{\mathbf{i}}(X, L^n \otimes H) = \cdots = \mathbf{h}^{\mathbf{i}}(X, L^n \otimes H^{m_n}) = 0 \quad \text{for all } i \geq 2.$$

For each $n \geq n_0$ the Riemann-Roch theorem gives

$$\mathbf{h}^{\mathbf{0}}(X, L^n) = \mathbf{h}^{\mathbf{1}}(X, L^n) + \frac{\mathbf{c}_{\mathbf{1}}^d(L^n)}{d!} + (\text{terms containing strictly lower powers of } \mathbf{c}_{\mathbf{1}})$$

and so, since $\mathbf{c}_{\mathbf{1}}^d(L) > 0$, we get that there exists an n big enough s.t. $\mathbf{h}^{\mathbf{0}}(X, L^n) > 0$. Q.E.D.

3. The class of the diagonal and of $\Gamma_n(g^r_d)$'s

We recall some theory from [A-C-G-H]. Consider the diagonal map

$$\phi_a = \phi \colon C_{d-2} \times C \to C_d$$

defined by

$$\phi(D, p) = D + 2p .$$

The image of this map is the diagonal Δ_d in C_d. A special case of Proposition 5.1 on p. 358 in [A-C-G-H] gives that

Lemma 7 (MacDonald). *The class δ_d of the diagonal Δ_d in C_d is given by*

$$\delta_d = 2((d + g - 1)x - \theta) . \tag{5}$$

We denote now by g^r_d a base point free linear system of degree d and dimension r on C. Given such a g^r_d, then for each n with $r < n \leq d$, we can construct in C_n the following cycle

$$\Gamma_n(g^r_d) = \{D \in C_n \text{ s.t. } D \leq E \text{ for some } E \in g^r_d\} .$$

The standard way to calculate the class $\gamma_n(g^r_d)$ of the above cycle is given by the following lemma, see [A-C-G-H, Lemma 3.2, p. 342].

Lemma 8. *For integers $d \geq n > r$ the class $\gamma_n(g^r_d)$ in C_n is given by*

$$\gamma_n(g^r_d) = \sum_{k=0}^{n-r} \binom{d-g-r}{k} \frac{x^k \theta^{n-r-k}}{(n-r-k)!} .$$

In the particular case where $n = r+1$ and so, $\Gamma_{r+1}(g^r_d)$ is a divisor in C_{r+1}, we can find the class as following:

We denote by $C^{\times(r+1)}$ the $(r+1)$th Cartesian product of C, by $f_1, \dots, f_{r+1}$ the class of the coordinate planes and by δ_C the class of the sum of the diagonals in the product. Also we define $\gamma_C = \pi^*(\gamma_{r+1}(g^r_d))$, where $\pi \colon C^{\times(r+1)} \to C_{r+1}$ the canonical map. We have the following relations:

$$\pi^* x = f \overset{\text{def}}{=} f_1 + \cdots + f_{r+1}, \quad \pi^*\delta_{r+1} = 2\delta_C, \quad \delta_{r+1} = 2((g+r)x - \theta) .$$

Given a g^r_d on C we have a canonical map $\phi \colon C \to \mathbf{P}^r = \mathbf{P}$ and an induced (product) map $\Phi \colon C^{\times(r+1)} \to \mathbf{P}^{\times(r+1)}$. We denote by $\delta_{\mathbf{P}}$ the class of the sum of the diagonals in $\mathbf{P}^{\times(r+1)}$. Observe that

$$\Phi^*(\delta_{\mathbf{P}}) = \delta_C + \gamma_C .$$

We have $\delta_{\mathbf{P}} = f_1^{\mathbf{P}} + \cdots + f_{r+1}^{\mathbf{P}}$, where $f_i^{\mathbf{P}}$'s are the classes of the coordinate planes in $\mathbf{P}^{\times(r+1)}$. Therefore, $\Phi^*(\delta_{\mathbf{P}}) = \Phi^*(f_1^{\mathbf{P}} + \cdots + f_{r+1}^{\mathbf{P}}) = d(f_1 + \cdots + f_{r+1}) = df$ and so, $\gamma_C = df - \delta_C$.

Now, $\pi^*(\gamma_{r+1}(g^r_d)) = \gamma_C = df - \delta_C = d\pi^* x - \frac{1}{2}\pi^*(\delta_{r+1})$ and so, $\gamma_{r+1}(g^r_d) = dx - \delta_{r+1}/2$. Using relation (5) we conclude that

$$\gamma_{r+1}(g^r_d) = \theta - (g - d + r)x . \tag{6}$$

4. First bounds for the cones

We examine the case $d \leq g$. If D is an effective divisor on C_d, then $u_d^* \theta^{d-1} \cdot D = \theta^{d-1} \cdot u_{d_*} D$, where u_d the Abel-Jacobi map. Since θ is an ample divisor on $J(C)$, we get by Lemma 3, that $\theta^{d-1} \cdot u_{d_*}D \geq 0$, where equality holds iff $u_{d_*}D = 0$. This gives the first naive bound for the effective cone in C_d:

Suppose that D is divisor with class $a\theta - bx$, $a, b > 0$ i.e. it "lies" in the fourth quarter of the θ, x-plane. We define slope m of D to be $m = \frac{b}{a}$.

If D is effective, then by the above discussion we have $(\theta - mx)\theta^{d-1} \geq 0$ which implies

$$\frac{g!}{(g-d)!} - m\frac{g!}{(g-d+1)!} \geq 0 \quad \text{i.e.} \quad m \leq g - d + 1.$$

If D is ample, Lemma 3 implies that $(\theta - mx)^d > 0$. Equivalently

$$\sum_{k=0}^{d} \binom{d}{k} \theta^k m^{d-k} x^{d-k} > 0,$$

i.e.

$$\sum_{k=0}^{d} \binom{d}{k} m^{d-k} \frac{g!}{(g-k)!} > 0. \tag{7}$$

Since for $m = 0$ this is positive, we must have $m <$ (min. posit. root of (7)).

For a divisor D with class $ax - b\theta$, $a, b > 0$, i.e. it "lies" in the second quarter of the θ, x-plane we define the slope $\overline{m}$ of D to be $\overline{m} = \frac{a}{b}$. Similar argument gives that if D is effective then $\overline{m} \geq g - d + 1$, and if D is ample then $\overline{m}$ satisfies a similar relation as in (7). For example if D is ample then for

$$d = 2 \text{ we have } m < g - \sqrt{g} \text{ and } \overline{m} > g + \sqrt{g}.$$

5. Effective and ample cones for C_2

By the previous discussion we have the first bounds for the effective and ample cones for C_2. On the other hand using the Lemma 4 we know that every

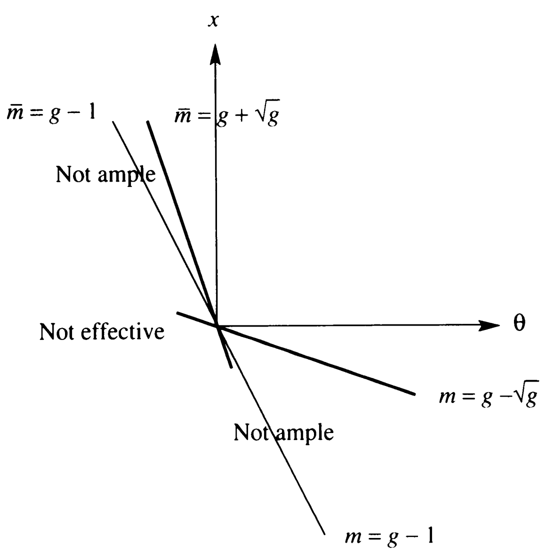

Figure 1. Bounds for the cones in C_2

class between the thick lines in Figure 1 is effective. Since the ample and effective cones are dual, it is enough to describe the effective cone. We make the observation:

Observation. If D is an irreducible effective divisor with slope m (resp. $\overline{m}$) between

$$g-\sqrt{g}<m\leq g-1 \quad (\text{resp. } g+\sqrt{g}>\overline{m}\geq g-1),$$

then it is unique with this property.

Indeed, if D' is another irreducible effective divisor with slope m' (resp. $\overline{m}'$) in the above range then we get $DD'<0$, a contradiction.

If we are able to find such a divisor, then this describes the cone. In the second quarter such a divisor exists, namely the diagonal: Recall that $\delta_2 = 2((g+1)x-\theta)$ and so $\overline{m}=g+1$ i.e. $g+\sqrt{g}>\overline{m}>g-1$. Therefore the slope of the effective cone in the second quarter is given by

$$\overline{m}_{ef}=g+1 \tag{9}$$

and (by duality) of the ample cone by

$$\overline{m}_a=2g\,. \tag{10}$$

(Note that a divisor with slope $\overline{m}_a$, has positive intersection with all the effective divisors in the first or the fourth quarter.)

Also, if C is of genus 2, then it is hyperelliptic and the corresponding g^1_2 gives an effective class in C_2 belonging in the fourth quarter with slope $m=1$, see formula (6). By the above observation we get that $m_{ef}=1$ and by duality $m_a=0$. For general genus, we have the following:

Theorem 2. *Let C be a curve of genus $g\geq 3$ with general moduli. For the slopes of the cones in C_2, in the fourth quarter of the θ, x-plane, we have*

1. *If g is a square, then $m_{ef}=m_a=g-\sqrt{g}$.*
2. *If g is not a square and $g>3$, then $g-[\sqrt{g}]+1\geq m_{ef}\geq g-\sqrt{g}$ and so*

$$g-\sqrt{g}\geq m_a\geq g-\frac{g}{[\sqrt{g}]-1}.$$

3. *If $g=3$, then $m_{ef}=\frac{4}{3}$ and $m_a=\frac{6}{5}$.*

Proof. We use degenerations to special curves: From the formula (6) we have that the class of $\gamma_2(g^1_d)$ in C_2 is given by

$$\gamma_2(g^1_d)=\theta-(g-d+1)x\,.$$

Therefore if $d<\sqrt{g}+1$, the slope of the divisor is

$$m^0=g-d+1\geq g-\sqrt{g}\,.$$

Take a smooth curve C^0 with a g^1_d, $d<\sqrt{g}+1$. If in addition, we choose the curve to be "general" having such a g^1_d, then the corresponding divisor $\Gamma_2(g^1_d)$ is irreducible in C_2. Since C^0 is special, the $H^*_{an}(C^0,\mathbf{Q})$ may not be generated by x, θ. Consider in $H^*_{an}(C^0,\mathbf{Q})$ the plane Π spanned by x, θ. By the previous analysis, since $m^0>g-\sqrt{g}$ the intersection of the effective cone with the plane Π is given by the slopes $\overline{m}^0_{ef}=g+1$ and $m^0_{ef}=g-d+1$.

Since $\mathscr{M}_g$ is connected, we can find a flat family of smooth curves $\mathscr{C}\to\Delta$, Δ a disk, with central fiber C^0 and the other fiber curves with general moduli.

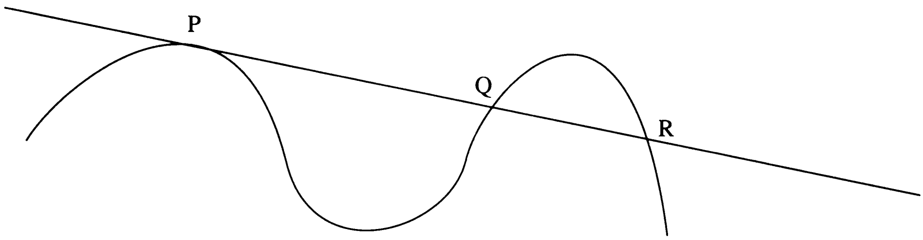

FIGURE 2. The genus 3 case

Let $\Phi\colon \mathscr{C}_2 \to \Delta$ be the flat family with fibers the 2-symmetric products of the fibers of $\mathscr{C} \to \Delta$.

Suppose that the general curve has an irreducible divisor with slope m, $g-1 \geq m > g-\sqrt{g}$. By the above observation, this is unique and so, it gives rise to an effective divisor in $\mathscr{C}_2 \backslash \Phi^{-1}(0)$. Degenerating to the central fiber C_2^0, we get an effective divisor D^0. Since the degeneration "preserves" the algebraic equivalence we have that D^0 belongs to Π and so, since it is effective, the slope m satisfies $g-d+1 \geq m \geq g-\sqrt{g}$.

Therefore if g is a square i.e. $g = k^2$, then choosing $d = k-1 = \sqrt{g}-1$ we get that m has to satisfy $m \leq g - \sqrt{g}$.

Since for slopes smaller than $g-\sqrt{g}$ we are in the effective cone by Lemma 4, we conclude that

$$m_{ef} = m_a = g - \sqrt{g}.$$

If g is not a square, choosing $d = [\sqrt{g}]$ we have

$$g - \sqrt{g} \leq m_{ef} \leq g - [\sqrt{g}] + 1.$$

The estimation of m_a comes from the duality of the ample and effective cone. Of course as $g \to \infty$ we have $m_{ef} \sim m_a \sim g - \sqrt{g}$. Q.E.D.

The case $g = 3$. In this case we can obtain for the general curve an irreducible divisor with slope $\frac{4}{3}$. (See Figure 2.) It is known that a general smooth curve C of genus 3, can be represented as a smooth plane quartic. We construct the following divisor in C_2: For each point P in C consider the tangent at that point. This intersects the curve at two additional points Q, R. Take now in C_2 the divisor D consisting of all the sums Q+R (with P moving on C). In order to calculate the class of D, we find its intersections with x and δ_2.

Intersection with x. The degree of the dual curve is 12. Fixing a point Q on C, we ask for the number of tangents to the curve passing through Q (excluding the tangent to the curve at Q). These are 10. So $Dx = 10$.

Intersection with δ_2. This is twice the number of bitangents to C, so $D\delta_2 = 28 \times 2 = 56$.

Therefore if $D \sim a\theta - bx$ then, $Dx = 10$ i.e. $3a - b = 10$ and $D\theta = 56$ i.e. $2(a\theta - bx)(4x - \theta) = 56$ so $6a - b = 28$. This gives $a = 6$, $b = 8$, i.e. $m_{ef} = \frac{4}{3}$, $m_a = \frac{6}{5}$.

6. ONE SIDE SLOPE OF EFFECTIVE CONE IN C_d

Theorem 3. *The boundary of the effective cone in* C_d, *in the second quarter of the* θ, *x-plane, is given by the class* δ_d *of the diagonal* Δ_d

$$\delta_d = 2((d+g-1)x - \theta) \quad \text{i.e.} \quad \overline{m}_{ef} = d + g - 1.$$

Proof (Induction on d). For $d = 2$ it has been proved. Assuming it is true for d, we prove it for $d+1$, i.e. we prove that in C_{d+1}, $\overline{m}_{ef} = d + g$.

Suppose that there exists an irreducible divisor D on C_{d+1} with class $mx_{d+1} - \theta_{d+1}$ and $m < d + g$ (we add subindices for avoiding confusion). Fixing a point in C, we define the canonical embedding $i_k \colon C_k \to C_{d+1}$. Note that $i_k^*(x_{d+1}) = x_k$ and $i_k^*(\theta_{d+1}) = \theta_k$. The image of C_d is an ample divisor on C_{d+1}, see [A-C-G-H, p. 310] and so, $i_d^*(D)$ is effective nonzero on C_d. Therefore the slope must satisfy $m \geq d + g - 1$.

Since Δ_{d+1}, D are irreducible the intersection $\Delta_{d+1}D$ is nonempty effective. Indeed, D, Δ are not disjoint. Otherwise,

$$m(d+g)x_{d+1}^2 - (m+d+g)\theta_{d+1}x_{d+1} + \theta_{d+1}^2 = 0.$$

Applying i_2^* we get $m(d+g)x_2^2 - (m+d+g)\theta_2 x_2 + \theta_2^2 = 0$. Then formula (1) implies $md = (d+1)g$. Since $i_2^*(\Delta_{d+1})$ contains Δ_2, Δ_2 must be disjoint from $i_2^*(D) = D_2$. Therefore, $\Delta_2 D_2 = 0$ i.e. $(mx_2 - \theta_2)\delta_2 = 0$ i.e. $m = 2g$, and the above relation becomes $2gd = (d+1)g$ i.e. $d = 1$ a contradiction.

Therefore $B_1(D\Delta_{d+1})$, see §1 for definition of B_1, lies in the effective cone of C_d, i.e.

$$\text{slope of } B_1(D\Delta_{d+1}) \geq d + g - 1. \tag{11}$$

Now,

$$\operatorname{class}(D\Delta_{d+1}) = m(d+g)x_{d+1}^2 - (m+d+g)x_{d+1}\theta_{d+1} + \theta_{d+1}^2.$$

By Lemma 2 we have

$$B_1(x_{d+1}^2) = 2x_d, \quad B_1(x_{d+1}\theta_{d+1}) = \theta_d + gx_d, \quad B_1(\theta_{d+1}^2) = 2(g-1)\theta_d.$$

Therefore,

$$B_1(D\Delta_{d+1}) = (2md + mg - dg - g^2)x_d - (m + d - g + 2)\theta_d.$$

Since $m \geq d + g - 1$, both coefficients are positive and so,

$$\text{slope of } B_1(D\delta_{d+1}) = \frac{2md + mg - dg - g^2}{m + d - g + 2}.$$

Relation (11) implies that

$$\frac{2md + mg - dg - g^2}{m + d - g + 2} \geq d + g - 1$$

i.e. $m(d+1) \geq dg + d^2 + d + 3g - 2$. Since $m < d + g$ we get $(d+g)(d+1) > dg + d^2 + d + 3g - 2$ or $1 > g$, a contradiction. Q.E.D.

7. Bounds for the effective cone in C_r, $r \geq 3$

We start with C_3. Let D be a divisor in C_3 with class $\theta - mx$, $m \geq 0$, i.e. it "lies" in the fourth quarter. Since x is the class of $i_2(C_2)$ in C_3 and

$$\frac{d}{dm}(\theta - mx)^3 = -3(\theta - mx)^2 x,$$

we conclude that

$$\frac{d}{dm}(\theta - mx)^3 = 0 \text{ in } C_3 \quad \Leftrightarrow \quad (\theta - mx)^2 = 0 \text{ in } C_2,$$

i.e. when $m = g + \sqrt{g}$ or $m = g - \sqrt{g}$.

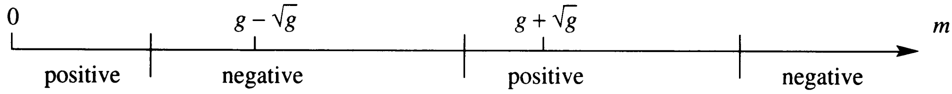

FIGURE 3. The graph of $(\theta - mx)^3$

The graph of the $(\theta - mx)^3$ considered as function of m is given by Figure 3.

Theorem 4. *We denote by r_1 the root of $(\theta - mx)^3 = 0$ closest to 0. Then for $m < r_1$, the class $\theta - mx$ is effective.*

Proof. Note that this equation has three positive roots, see Figure 3 above. Approximately as $g \to \infty$, r_1 goes to $g - \sqrt{3g}$. Say D a divisor with class $\theta - mx$ where $m < r_1$. By Theorem 2 the restriction of D to C_2 is ample. Since $D^3 > 0$, Theorem 1, applied to $H = C_2$, gives the result. Q.E.D.

It is difficult to continue the above method for higher r's, since we do not have a good estimate for the ample cone in C_3. For these cases we have

Theorem 5. *Let C be a smooth curve of genus $g \geq 1$ with general moduli. For $r \geq 3$ we have the following estimates for the effective and ample cone of C_r in the fourth quarter of the θ, x-plane*

1. *If $r \geq g + 1$ then the boundary of the effective and ample cone is given by the θ line.*

2. *If $3 \leq r \leq g$ then we have the following bounds for the effective cone.*

Bound from inside: for each rational m with

$$0 \leq m \leq \max([\tfrac{g}{r}], g - (r-1)\sqrt{g})$$

there is an effective divisor with slope m.

Bound from outside: for each m with $m > g - [\sqrt{r-1}\sqrt{g}]$ there is no effective divisor with slope m.

Remarks. 1. For $r = g$ the slope of the boundary of the effective cone is equal to 1. Indeed in this case $[\frac{g}{r}] = 1 = g - [\sqrt{r-1}\sqrt{g}]$.

2. For $r \leq \sqrt{g}$ we have that

$$\max([\tfrac{g}{r}], g - (r-1)\sqrt{g}) = g - (r-1)\sqrt{g}.$$

For $r \geq \sqrt{g}$ we have that

$$\max([\tfrac{g}{r}], g - (r-1)\sqrt{g}) = [\tfrac{g}{r}].$$

Proof. The proof of the first part of the theorem is easy: For $r \geq g + 1$ we have that $\theta^r = u_r^*(\theta)^r = 0$ and so, the class θ is not ample. On the other hand, since θ is ample on the Jacobian and x is ample on C_r, see [A-C-G-H, p. 310], projection formula and Lemma 3, imply that the class $\theta + \varepsilon x$ is ample, for each $\varepsilon > 0$. Therefore the bound for the ample cone is given by the line θ. Also any divisor with class $\theta - \sigma x$ cannot be effective since there is an $\varepsilon > 0$ small enough with $(\theta + \varepsilon x)^{r-1}(\theta - \sigma x) < 0$; a contradiction by Lemma 3. Therefore the bound of the effective cone is given by the θ line too.

To prove the second part of the theorem we use again degenerations to special curves. We start with a lemma:

Lemma 9. *If a curve C has a "general" g_d^{r-1}, $r \geq 3$ (i.e. without base points and not composed with an involution), then for any $d \geq r$ the divisor $\Gamma_r(g_d^{r-1})$ in C_r is irreducible.*

Proof. This is an application of the fact that the monodromy acts as the full symmetric group on the generic divisor of g_d^r. Q.E.D.

Let us now do some calculations. Recall from relation (6), that the class of $\gamma_r(g_d^{r-1})$ is given by $\theta - (g-d+r-1)x$. Formula (1) gives that

$$x^k\theta^{r-k} = \frac{g!}{(g-r+k)!} \quad \text{and} \quad x^{k+1}\theta^{r-k-1} = \frac{g!}{(g-r+k+1)!}.$$

Using Lemma 8 we have for $d+1 \geq 2r$ that

$$\gamma_r(g^1_{d-r+2}) = \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \frac{x^k\theta^{r-k-1}}{(r-k-1)!}. \tag{12}$$

Therefore intersection number $I = \gamma_r(g_d^{r-1}) \cdot \gamma_r(g^1_{d-r+2})$ is

$$\begin{aligned}
I &= \sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \frac{g!}{(g-r+k)!(r-k-1)!} \\
&\quad - (g-d+r-1)\sum_{k=0}^{r-1} \binom{d-r-g+1}{k} \frac{g!}{(g-r+k+1)!(r-k-1)!} \\
&= g\sum_{k=0}^{r-1} \binom{d-r-g+1}{k}\binom{g-1}{r-k-1} \\
&\quad - (g-d+r-1)\sum_{k=0}^{r-1} \binom{d-r-g+1}{k}\binom{g}{r-k-1} \\
&= g\binom{d-r}{r-1} - (g-d+r-1)\binom{d-r+1}{r-1} \\
&= A(d^2 - 2d(r-1) - (r-1)(g-r-1)) \quad (A \text{ a positive constant}).
\end{aligned}$$

This implies that

$$I \leq 0 \Leftrightarrow r-1-\sqrt{r-1}\sqrt{g} \leq d \leq r-1+\sqrt{r-1}\sqrt{g}. \tag{13}$$

Take now a smooth curve C^0 having a "general" $g_{d_0}^{r-1}$ with $d_0 = r+1+[\sqrt{r-1}\sqrt{g}]$ (note that $d_0 \geq r$). We claim that the irreducible divisor $D = \Gamma_r(g_{d_0}^{r-1})$ with class $\gamma_r(g_{d_0}^{r-1}) = \theta - (g - [\sqrt{r-1}\sqrt{g}])x$ gives the bound for the effective cone in the θ, x-plane for $(C^0)_r$. Indeed, note first that the divisor D is covered by a family of curves $\mathscr{C}_{d_0-r+2}$ with class $\gamma_r(g^1_{d_0-r+2})$. These curves correspond to the various g^1_{d-r+2}'s obtained by the one-parameter family of hyperplane sections of the image of the curve in $\mathbf{P}^{r-1}$, through collections of $r-2$ fixed points on this curve. Suppose that there exists another irreducible effective divisor D' with slope m' strictly greater than the slope of D. Then relation (13) implies that $D' \cdot \gamma_r(g^1_{d_0-r+2}) < 0$ and so, since D' is irreducible we get that all the members of $\mathscr{C}_{d_0-r+2}$ are contained in D'. But since the divisor

D' is "covered" by the family $\mathscr{C}_{d_0-r+2}$ this implies that D is contained in D'. A contradiction. The rest of the proof for the bound from outside goes, using degenerations after a possible base change, as the proof of Theorem 2.

To prove the case for the bound from inside, we use the maps A_k defined in the introduction of this paper. From formula (2) we have

$$A_{r-2}(x_r) = (r-1)x_2 \quad \text{and} \quad A_{r-2}(\theta_r) = \theta_2 + g(r-2)x_2 . \tag{14}$$

Since in C_2 the slope m with $m < g - \sqrt{g}$ is "effective", pulling back by the A_{r-2} map we get in C_r that

$$A_{r-2}(\theta_r - (g-\sqrt{g})x_r) = \theta_2 - (g-(r-1)\sqrt{g})x_2 . \tag{15}$$

On the other hand we have that for $d = g+r-1-[\frac{g}{r}]$, a general curve has a g_d^{r-1} (this is the minimum d for which the Brill-Noether number ρ is nonnegative). Using formula (6) we obtain the following class of an effective divisor in C_r:

$$\gamma_r(g_d^{r-1}) = \theta - [\tfrac{g}{r}]x \tag{16}$$

and this concludes the proof. Q.E.D.

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